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Path integral quantisation of the open string's normal modes

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Abstract. The propagator for the bosonic string is obtained by making use of the bosonic dual resonance model. The two-stage gauge-fixing process employed circumvents the string's Gribov problem and verifies the integration measure found by Govaerts.

In this paper the bosonic string propagator is obtained using Govaerts' amended form of the Fradkin-Vilkovisky (FV) theorem [1, 2]. The derivation makes use of the dual resonance model via a two-stage gauge-fixing process and confirms the result obtained by Govaerts using a more direct method [3]. The principle advantage of the approach presented below is that the path integral measure emerges naturally from the formalism of Batalin, Fradkin and Vilkovisky (BFV) (see [1]). Govaerts makes the point that the string suffers from a Gribov problem; i.e. no good gauge-fixing functions exist and the path integral measure must therefore be determined by some indirect means.

Any point on the string world sheet can be parametrised by two coordinates; a timelike parameter τ and a spacelike parameter σ. In terms of these parameters the action (see [4]) for the massless relativistic string is

$$S=\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\pi}\mathrm{d}\sigma\,\mathscr{L}=-\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\int_0^{\pi}\mathrm{d}\sigma[(x'\dot{x})^2-(x')^2(\dot{x})^2]^{1/2} \qquad (1)$$

where $\dot{x}=\partial x/\partial\tau$ and $x'=\partial x/\partial\sigma$ and α' is a constant with the dimension of length squared. S is invariant under reparametrisations $\sigma\to\tilde{\sigma}(\sigma,\tau)$; $\tau\to\tilde{\tau}(\sigma,\tau)$ and local Weyl rescalings of the world sheet metric $g_{ab}\to\Lambda(\sigma)g_{ab}$. The dynamical variables are $x_\mu(\sigma,\tau)$ and $\bar{x}_\mu(\sigma,\tau)$. The momentum $\bar{x}_\mu(\sigma,\tau)$, conjugate to $x_\mu(\sigma,\tau)$, is defined by

$$\bar{x}_\mu(\sigma,\tau)=\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu(\sigma,\tau)} \qquad (2)$$

and the Poisson bracket

$$\{x_\mu(\sigma,\tau),\bar{x}^\nu(\sigma',\tau)\}_{\mathrm{PB}}=\delta^\nu_\mu\delta(\sigma-\sigma') \qquad (3)$$

is the only non-zero bracket. We find two primary constraints

$$T_1=\bar{x}_\mu\bar{x}^\mu+\frac{1}{4\pi^2\alpha'^2}x'_\mu x'^\mu\approx0 \qquad T_2=\bar{x}_\mu x'^\mu\approx0. \qquad (4)$$

The Poisson brackets between the constraints are weakly zero, so T_1 and T_2 are first class. The Hamiltonian density on the surface of constraint is zero and the consistency conditions $\dot{T}\approx0$ determine the total Hamiltonian density, $\mathscr{H}_{\mathrm{T}}$, without the occurrence of any further constraints:

$$\mathscr{H}_{\mathrm{T}}=\lambda_1(\sigma,\tau)T_1(\sigma,\tau)+\lambda_2(\sigma,\tau)T_2(\sigma,\tau) \qquad (5)$$

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where the $\lambda(\sigma, \tau)$ are Lagrange multipliers for the constraints. The total Hamiltonian is $H_T = \int_0^\pi d\sigma\, \mathcal{H}_T$. The Hamiltonian can be expressed in an alternative form by extending the domain of σ from $[0, \pi]$ to $[-\pi, \pi]$. Any linear combination of the constraints is itself a constraint and so

$$\bar{x}_\mu \bar{x}^\mu + \frac{1}{\pi\alpha'} \bar{x}_\mu x'^\mu + \frac{1}{4\pi^2\alpha'^2} x'_\mu x'^\mu \approx 0 \qquad \sigma \in [-\pi, \pi]. \tag{6}$$

Defining

$$\bar{x}_\mu(-\sigma) = \bar{x}_\mu(\sigma) \qquad x'_\mu(-\sigma) = -x'(\sigma) \tag{7}$$

the linear combination (6) gives back our two constraints T_1 and T_2 on the interval $\sigma \in [0, -\pi]$. H_T can then be expressed as

$$H_T = \int_{-\pi}^{\pi} d\sigma\, \lambda(\sigma, \tau) \left(\bar{x}_\mu \bar{x}^\mu + \frac{1}{\pi\alpha'} \bar{x}_\mu x'^\mu + \frac{1}{4\pi^2\alpha'^2} x'_\mu x'^\mu \right). \tag{8}$$

We can write the Hamiltonian in terms of the string's Fourier coefficients if we substitute into (8) the following even Fourier expansions for x_μ and $\bar{x}_\mu$.

$$\begin{aligned} x^\mu(\sigma, \tau) &= \frac{x_0^\mu(\tau)}{\sqrt{2}} + \sum_{n=1}^{\infty} x_n^\mu(\tau) \cos(n\sigma) \\ \bar{x}^\mu(\sigma, \tau) &= \frac{\bar{x}_0^\mu(\tau)}{\sqrt{2}} + \sum_{n=1}^{\infty} \bar{x}_n^\mu(\tau) \cos(n\sigma). \end{aligned} \qquad \sigma \in [-\pi, \pi] \tag{9}$$

(In making these expansions on the interval $[-\pi, \pi]$ we have implicitly assumed the open-string boundary conditions, $x'_\mu(0, \tau) = x'_\mu(\pi, \tau) = 0$.) With the Fourier expansions (9) and the gauge choice $d\lambda(\sigma, \tau)/d\sigma = 0$, H_T can be rewritten as

$$H_T = 2\lambda(\tau) \int_0^\pi d\sigma \left(\bar{x}^\mu \bar{x}_\mu + \frac{1}{4\pi^2\alpha'^2} x'^\mu x'_\mu \right). \tag{10}$$

By considering Hamilton's equations

$$\dot{x}^\mu(\sigma, \tau) = \{x^\mu(\sigma, \tau), H_T\}_{PB} \qquad \dot{\bar{x}}^\mu(\sigma, \tau) = \{\bar{x}^\mu(\sigma, \tau), H_T\}_{PB} \tag{11}$$

we obtain the equation of motion

$$\ddot{x}^\mu(\sigma, \tau) = \frac{4\lambda^2(\tau)}{\pi^2\alpha'^2} x''^\mu(\sigma, \tau). \tag{12}$$

This is the equation describing transverse oscillations of a string: in the gauge $d\lambda(\sigma, \tau)/d\sigma = 0$ only the tranverse modes of oscillation contribute to the string Hamiltonian H_T. Substituting equations (9) into (10) gives

$$H_T = \lambda(\tau)\pi \sum_{n=0}^{\infty} \left(\bar{x}_n^\mu(\tau) \bar{x}_{n\mu}(\tau) + \frac{n^2}{4\pi^2\alpha'^2} x_n^\mu(\tau) x_{n\mu}(\tau) \right). \tag{13}$$

Note that with the choice $\lambda(\tau) = 1/2\pi$, H_T becomes the Nambu Hamiltonian, H_B; see [5]. From [5] we have

$$\{x_m^\mu(\tau), \bar{x}_{n\nu}(\tau)\}_{PB} = \delta_\nu^\mu \delta_{mn} \qquad \{x_m^\mu(\tau), x_{n\nu}(\tau)\}_{PB} = 0 = \{\bar{x}_m^\mu(\tau), \bar{x}_{n\nu}(\tau)\}_{PB} \tag{14}$$

as the Poisson brackets for the Fourier coefficients.

We now quantise the string in the gauge $\mathrm{d}\lambda(\sigma,\tau)/\mathrm{d}\sigma=0$ using the BFV approach [1]. Since $H_0=0$, H_T must be a linear combination of constraints. From (13) this leads us to regard

$$T=\sum_{n=0}^{\infty}\left(\bar{x}_n^{\mu}(\tau)\bar{x}_{n\mu}(\tau)+\frac{n^2}{4\pi^2\alpha'^2}x_n^{\mu}(\tau)x_{n\mu}(\tau)\right)\approx 0 \tag{15}$$

as the only phase space constraint. T is therefore first class. The dynamical variables defining the theory are $x_n^{\mu}(\tau)$ and $\bar{x}_n^{\mu}(\tau)$. The multiplier λ is now taken as a dynamical variable with conjugate momentum ν. We set $\nu\approx 0$, so that the dynamics of the original theory remains unchanged. The phase space is further extended by introducing a fermionic ghost variable η and conjugate momentum $\bar{\eta}$ for the first-class constraint T. Similarly, we require fermionic ghosts ζ and $\bar{\zeta}$ for the conjugate multiplier ν. The superlarge phase space is

$$\{x_n^{\mu}, \bar{x}_{n\mu}, \lambda, \nu, \eta, \bar{\eta}, \zeta, \bar{\zeta}\} \qquad n=0,1,\ldots,\infty \tag{16}$$

with the Poisson brackets

$$\{\lambda,\nu\}_{\mathrm{PB}}=1 \qquad \{\eta,\bar{\eta}\}_{\mathrm{PB}}=-1 \qquad \{\zeta,\bar{\zeta}\}_{\mathrm{PB}}=-1 \tag{17}$$

in addition to equations (14). Writing $G_a=(\nu, T)$, the constraint algebra is

$$\{G_a, G_b\}_{\mathrm{PB}}=0 \qquad \{G_a, H_0\}_{\mathrm{PB}}=0. \tag{18}$$

The BRST generator is $\Omega=\nu\zeta+T\eta$ and the BFV path integral is

$$Z=\int[\mathrm{d}x_n^{\mu}][\mathrm{d}\bar{x}_{n\mu}][\mathrm{d}\lambda][\mathrm{d}\nu][\mathrm{d}\eta][\mathrm{d}\bar{\eta}][\mathrm{d}\zeta][\mathrm{d}\bar{\zeta}] \times\exp\left[\mathrm{i}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau\left(\sum_{n=0}^{\infty}\dot{x}_n^{\mu}\bar{x}_{n\mu}+\dot{\lambda}\nu+\dot{\eta}\bar{\eta}+\dot{\zeta}\bar{\zeta}-\{\Psi,\Omega\}_{\mathrm{PB}}\right)\right] \tag{19}$$

with the BRST invariant boundary conditions

$$\begin{aligned} &x_n^{\mu}(\tau_\mathrm{i})=x_{n\mathrm{I}}^{\mu} && x_n^{\mu}(\tau_\mathrm{f})=x_{n\mathrm{F}}^{\mu} && \\ &\nu(\tau_\mathrm{i})=0=\nu(\tau_\mathrm{f}) && \eta(\tau_\mathrm{i})=0=\eta(\tau_\mathrm{f}) && \bar{\zeta}(\tau_\mathrm{i})=0=\bar{\zeta}(\tau_\mathrm{f}). \end{aligned} \tag{20}$$

The gauge choice $\Psi=-\lambda\bar{\eta}$ gives

$$Z=\int[\mathrm{d}x_n^{\mu}][\mathrm{d}\bar{x}_n^{\mu}][\mathrm{d}\lambda][\mathrm{d}\nu][\mathrm{d}\eta][\mathrm{d}\bar{\eta}][\mathrm{d}\zeta][\mathrm{d}\bar{\zeta}] \times\exp\left[\mathrm{i}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau\left(\sum_{n=0}^{\infty}\dot{x}_{n\mu}\bar{x}_n^{\mu}+\dot{\lambda}\nu+\dot{\eta}\bar{\eta}+\dot{\zeta}\bar{\zeta}-\lambda T+\bar{\eta}\zeta\right)\right]. \tag{21}$$

Integrating over ν yields a delta functional in $\dot{\lambda}$; the above gauge choice is a proper-time gauge. Integrating over λ and the ghost variables gives

$$Z\sim\int_{-\infty}^{\infty}\mathrm{d}\lambda_0(\tau_\mathrm{f}-\tau_\mathrm{i})\int[\mathrm{d}x_0^{\mu}][\mathrm{d}\bar{x}_0^{\mu}]\exp\left(\mathrm{i}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau(\dot{x}_0^{\mu}\bar{x}_{\mu}^0-\lambda_0\bar{x}_0^{\mu}\bar{x}_{\mu}^0)\right)\prod_{n=1}^{\infty}K_n(\lambda_0) \tag{22}$$

where

$$K_n(\lambda_0)=\int[\mathrm{d}x_n^{\mu}][\mathrm{d}\bar{x}_n^{\mu}]\exp\left(\mathrm{i}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau[\dot{x}_{n\mu}\bar{x}_n^{\mu}-\lambda_0(\bar{x}_{n\mu}\bar{x}_n^{\mu}+\omega_n^2x_{n\mu}x_n^{\mu})]\right) \tag{23}$$

where $\omega_n = n/2\pi\alpha'$ and λ_0 is a (τ-independent) constant. Equation (22) shows the string propagating as a product of an infinite number of independent normal modes. The zero-mode contribution has the same form as the massless point-particle propagator (see [2]). K_n is a harmonic oscillator path integral and can be evaluated by applying th standard result in [6] giving

$$K_n = \left(\frac{\omega_n}{2\pi \mathrm{i}\sin(2\lambda_0\omega_n(\tau_f-\tau_i))}\right)^{D'/2} \times\exp\left(\frac{\mathrm{i}\omega_n[(x^{\mathrm{F}}_{n\mu}x^{n\mu}_{\mathrm{F}}+x^{\mathrm{I}}_{n\mu}x^{n\mu}_{\mathrm{I}})\cos(2\lambda_0\omega_n(\tau_f-\tau_i))-2x^{\mathrm{F}}_{n\mu}x^{n\mu}_{\mathrm{I}}]}{2\sin[2\lambda_0\omega_n(\tau_f-\tau_i)]}\right) \tag{24}$$

where D' is the number of string mode polarisations. Since the timelike and longitudinal modes do not contribute to the string Hamiltonian in the gauge $\mathrm{d}\lambda(\sigma,\tau)/\mathrm{d}\sigma = 0$ we set $D' = D-2$ where D is the dimension of spacetime.

Since the $\lambda(\sigma,\tau)$ are invariant under Weyl rescalings a good gauge slice of the space of world sheet metrics corresponds to a good gauge slice of the space of multipliers. With $H_0 = 0$ the string's Teichmuller space can be written as

$$\mathrm{Teich} = \frac{\{\lambda(\sigma,\tau)\}}{\{\mathrm{rescalings}\}\times\{\mathrm{reparametrisations}\}}. \tag{25}$$

Having integrated over σ, the reparametrisation under which the action is invariant is now $\tau\to f(\tau)$ so that we can write

$$\mathrm{Teich} = \frac{\{\lambda(\tau)\}}{\{\tau\to f(\tau)\}}. \tag{26}$$

The parameter $c = \int_{\tau_i}^{\tau_f}\mathrm{d}\tau\,\lambda(\tau)$ is now the Teichmuller parameter for the string. In a proper time gauge, $c = \lambda_0(\tau_f-\tau_i)$. Writing Z in terms of the Teichmuller parameter gives

$$Z \sim \int_{-\infty}^{\infty}\mathrm{d}c\left(\frac{\pi}{\mathrm{i}c}\right)^{D/2}\exp\left(\frac{\mathrm{i}(x^0_{\mu\mathrm{F}}-x^0_{\mu\mathrm{I}})^2}{4c}\right)\prod_{n=1}^{\infty}K_n(c) \tag{27}$$

where we have quoted the closed form for the zero-mode propagator from [2]. Notice that the action (1) is invariant under the diffeomorphism $\tau_i\leftrightarrow\tau_f$, $\mathrm{d}\tau\to-\mathrm{d}\tau$. So, as for the point particle, the modular group is $\mathbb{Z}_2$ and

$$\text{Moduli space} = \frac{\mathrm{Teich}}{\mathbb{Z}_2}. \tag{28}$$

$\mathbb{Z}_2$ maps c into $-c$, so the following path integral, obtained from (27) by restricting the integration over c to a single cover of moduli space, is the string propagator:

$$Z \sim \int_0^{\infty}\mathrm{d}c\left(\frac{\pi}{\mathrm{i}c}\right)^{D/2}\exp\left(\frac{\mathrm{i}(x^0_{\mu\mathrm{F}}-x^0_{\mu\mathrm{I}})^2}{4c}\right)\prod_{n=1}^{\infty}\left(\frac{\omega_n}{2\pi\mathrm{i}\sin(2c\omega_n)}\right)^{(D-2)/2} \times\exp\left\{\frac{\mathrm{i}\omega_n((x^{\mathrm{F}}_{n\mu}x^{n\mu}_{\mathrm{F}}+x^{\mathrm{I}}_{n\mu}x^{n\mu}_{\mathrm{I}})\cos(2c\omega_n)-2x^{\mathrm{F}}_{n\mu}x^{n\mu}_{\mathrm{I}})}{2\sin(2c\omega_n)}\right\}. \tag{29}$$

The boundary conditions on the timelike and longitudinal modes must be $x_{n\mathrm{F}} = x_{n\mathrm{I}} = 0$.

By zeta function regularisation [3] we can replace

$$\prod_{n=1}^{\infty}\left(\frac{\omega_n}{2\pi\mathrm{i}\sin(2c\omega_n)}\right)^{(D-2)/2}$$

by

$$\prod_{n=1}^{\infty}\left(\frac{\pi}{\omega_n}[1-\exp(-4\mathrm{i}c\omega_n)]\right)^{-(D+2)/2}\frac{\exp[\mathrm{i}c(D-2)]}{24\pi\alpha'}.$$

The factor $\exp[\mathrm{i}c(D-2)]/24\pi\alpha'$ is the contribution to Z from the zero-point energy of the string's normal modes. If m_0 is the string's ground-state mass we have (see [7])

$$m_0^2=\frac{2-D}{24\alpha'}=\frac{-1}{\alpha'}\Rightarrow D=26. \tag{30}$$

The usual statement of the Fradkin-Vilkovisky theorem is that the BFV path integral (in this case (19)) is independent of the choice of Ψ. Govaerts presents counter examples to this orthodox statement for the case of the point particle [2]. He restates the FV theorem by saying that the path integral does not depend specifically on Ψ but on the gauge equivalence class of Ψ. Two gauge-fixing functions Ψ and Ψ' belong to the same equivalence class if they have identical Dirac brackets with the BRST generator Ω. Any two gauge equivalent Ψ will lead to the same covering of moduli space in the path integral. The correct propagator should be obtained as an integral over a single covering of moduli space. The equivalence class of the Ψ which lead to such a single covering of moduli space is referred to as the class of good gauge-fixing functions. In Govaerts' work [3] on the string path integral he conjectures that no good gauge-fixing functions exist for the string. That is, the string suffers from a Gribov problem. However, with the above two-stage gauge-fixing approach we do obtain Z as an integral over a single covering of moduli space.

Govaerts makes the point that a complete gauge fixing for the string is achieved by the conditions

$$\lambda_1(\sigma,\tau)=\lambda_0 \qquad \lambda_2(\sigma,\tau)=\lambda_0. \tag{31}$$

At least three equations are required to completely specify this gauge fixing:

$$\lambda_1(\sigma,\tau)=\lambda_2(\sigma,\tau)=\lambda(\sigma,\tau) \qquad \frac{\partial\lambda(\sigma,\tau)}{\partial\sigma}=0 \qquad \frac{\partial\lambda(\sigma,\tau)}{\partial\tau}=0. \tag{32}$$

It is explained in [3] that the number of these equations provided by the choice of Ψ is equal to the number of multipliers $\lambda(\sigma,\tau)$. These conditions emerge as delta functionals in the gauge-fixed BFV path integral. Since the BFV formalism gives only two Lagrange multipliers for the string, the choice of Ψ cannot give a complete gauge fixing. This is a clear statement of the string's Gribov problem. It can now be seen that the two-stage gauge fixing presented here gets around the Gribov problem by piecing together the required three equations in three distinct steps. In combining T_1 and T_2 to give a single constraint T we implicitly assume that $\lambda_1(\sigma,\tau)=\lambda_2(\sigma,\tau)$. $(\partial/\partial\sigma)\lambda(\sigma,\tau)=0$ was then specified in order to obtain the normal mode picture of the string (13). With a single multiplier remaining, the gauge choice $\Psi=-\lambda\bar{\eta}$ delivers the final equation as the delta functional $\delta[\dot{\lambda}]$, giving a complete gauge fixing. (Note that a two-stage gauge fixing was used by Narain [8] to circumvent the Gribov problem in the path integral quantisation of monopole strings.)

Govaerts gets round the string's Gribov problem by over-fixing the gauge, so that the covering of moduli space reduces to a single point, and then integrating over moduli space with an undetermined function $\mu(c)$ inserted. A careful comparison of the above propagator with [3] shows that the two methods are essentially the same. Govaerts' integral over the non-zero modes of the multipliers and ghosts exactly cancels the contributions from the unphysical longitudinal and timelike modes of the string coordinates. In the above approach, consideration of the classical equation of motion (12) shows that the unphysical modes do not contribute to Z. Govaerts' eventual conclusion that $\mu(c) = 1$ does not follow naturally from the path integral formalism, but by comparison with the point-particle path integral. In the above, the propagator is obtained as an integral over moduli space such that $\mu(c) = 1$ is automatically verified.

Killingback [9] gives a geometrical description of the Gribov problem. He explains that it is not possible to find a gauge for the string path integral which is valid over the whole of moduli space. Local gauges can, however, always be found. It is, then, surprising to obtain $\mu(c) = 1$ in the above: this gives the impression that the Gribov problem does not cause any difficulties in the string path integral after all. Fabbrichesi [10] has suggested that the Gribov problem corresponds to a particular BRST transformation and so is of no consequence in the BFV path integral. The specific meaning of this statement is not, however, clear to me. It is possible that assuming the Fourier expansions (9) (as Govaerts does also) gives a restriction to a region of moduli space over the whole of which a single gauge fixing is valid.

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